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Random searching

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Abstract

There are a wide variety of searching problems from molecules seeking receptor sites to predators seeking prey. The optimal search strategy can depend on constraints on time, energy, supplies or other variables. We discuss a number of cases and especially remark on the usefulness of Lévy walk search patterns when the targets of the search are scarce.

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1. A real search problem

Morse and Kendall [1] were involved in operations research during World War II and recounted one of their projects involving hunting for submarines. They introduced the operational quantity Q , measured in square miles per day,

$$Q = \frac{AC}{NT}, \quad (1)$$

where C is the number of contacts in a time T , A is the area surveyed and N is the number of submarines in A . They also have a theoretical expression for Q ,

$$Q = 2Rv, \quad (2)$$

where R is the detection range of the instruments and v is the search speed. Aircraft and ships will have different Q values. Optimal Q values were sought and better training and changes in tactics could raise the Q value. A following drop in the Q value most likely meant that the enemy had adapted to your new strategy. Questions arose, such as, if the Q value was higher near the shore, did this imply that more enemy submarines were deployed near the shore? Or it could mean that more patrols were executed near the shore, but more submarines were deployed in deeper waters. Complications arise because an unsuccessful search might result from a submarine being submerged and undetected, so areas had to be repeatedly searched. Submarines could be outfitted to detect a ship's or aircraft's radar and the

submarine could then take the appropriate evasive maneuvers. Aircraft adapted by lowering the intensity of their radar when a contact was made to trick the submarine into believing that the aircraft was leaving the area. This real-world search illustrates the point-counterpoint strategies in predator-prey relations. A random walk search would not make sense here, but such search modalities do appear naturally at the molecular level when diffusion processes dominate.

2. A diversity of search problems

Search problems cover a wide range of possibilities. In submarine warfare, it was a constant game of tactics and countermeasures based on advances in learning and implementing new technology. In animals a similar path is taken based on learning and over the long haul, natural selection. At the molecular level, in the brain, it is crucial for neurotransmitters (e.g. dopamine, serotonin) to find binding sites on transmembrane proteins [2]. The underlying physics for binding relies on hydrophobic and electrical forces and the particular molecular structure of the binding site. Proteins reaching binding sites on rapidly folding DNA combine 1D diffusion along the chain and longer jumps associated with chain looping [3, 4]. Some animals are ambush predators and wait motionlessly for prey to cross their path. Others, like the albatross, fly long distances to find food, but make use of the wind to glide and thus expend little energy. Benichou *et al* [5, 6] combine alternating diffusion with a static ambush-type capture. In salmon runs, bears know exactly where to find the salmon; the search question is only one of showing up at the right time. Tigers are lone predators, while army ants search large areas as a group. In this paper, we will study the effects of random searching for random walks with asymptotic Gaussian behavior and those with asymptotic fractal Lévy behavior [7, 8]. Good data is scarce and an expert knowledge is required for its analysis. The albatross data for the histogram of time spent in the air between landings once thought to be scale invariant Lévy-like now appears to be the more familiar exponential decay [9]. Actual search patterns of animals will depend on many factors: amount of energy expended in different modes of travel [8]; the probability of finding food during various locomotions (flying, running, walking, hopping, etc); whether a single animal or a group is executing the search [10]; day or night conditions; topography; weather; fixed food sources (water and vegetation) or moving targets (prey) [8]; homogeneous or scarce food sources [11]; whether the animal randomly searches for food or has knowledge of food locations.

A Lévy search strategy (never staying in one region for too long) might simply make a predator less of a target to its own enemies.

3. The gambler's ruin (a search to end a game)

A successful random search is related to the first passage time probability problem. The *first* first passage time calculation arose in the gamblers ruin problem. In this problem two gamblers win or lose a coin in each round of play. Jacob Bernoulli in his *Ars Conjectandi* (Art of Conjecture) published posthumously in 1713 (Jacob had died in 1705) calculated the probability that player A with n coins would win all of m coins of player B. For equal probability for A or B to win a round the probability that player A eventually wins all of the coins is $n/(n + m)$. This is also the probability of the ruin of player B. The result is more complicated if the A and B have different probabilities for winning a round. DeMoivre derived that the average duration of the game scales as n^2 if both players start with n coins. While a random search in $1d$ would not be optimal, it can apply to a diffusion process in a channel.

4. Number of sites visited in a set

The gamblers ruin duration n^2 result in $1d$ is connected to an n -step random walk in $1d$ visiting a number of distinct sites proportional to $n^{1/2}$. The formulas for a number of distinct sites, S_n , visited in a nearest-neighbor n -step random walk are [12]

$$S_n \approx \begin{cases} (8n/\pi)^{1/2} & 1d \\ \pi n / \log n & 2d \\ n/1.5164 & 3d \end{cases} \quad (\text{cubic lattice}). \quad (3)$$

These results follow from the basic equation connecting the probability $P_n(\ell)$ for being at site ℓ at the n th step and $f_n(\ell)$ the probability for being at site ℓ for the first time at the n th step,

$$p_n(\ell) = \sum_{j=1}^n f_j(\ell) p_{n-j}(0) + \delta_{n,0} \delta_{\ell,0}. \quad (4)$$

Multiplying by z^n and summing over all n , this equation takes the generating function form

$$F(\ell, z) = \frac{P(\ell, z) - \delta_{\ell,0}}{P(\ell=0, z)}, \quad (5)$$

where $P(\ell, z) = \sum_{n=1}^{\infty} p_n(\ell) z^n$ and $F(\ell, z) = \sum_{n=1}^{\infty} f_n(\ell) z^n$. DeMoivre also used recursion relations and generating functions to solve the gamblers ruin problem. Laplace applied generating functions, in the form of Laplace transforms, to the solution of probability problems.

One can choose a subset of the lattice points and study how many points of the subset are visited in an n -step random walk. For example, in $2d$, the number of points visited on a line m units from the random walk origin and in, $3d$, the number of points visited on a plane are given by [13]

$$\begin{aligned} S_{\text{line}}(n) &\approx (8\pi n)^{1/2} / \ln n & 2d \\ S_{\text{line}}(n) &\approx \ln n / P(\ell=0, z=1) & 3d \\ S_{\text{plane}}(n) &\approx (2n/\pi)^{1/2} / P(\ell=0, z=1) & 3d. \end{aligned} \quad (6)$$

These results are not the best search strategy, but a statement of the results for a random walk search.

5. Photosynthetic units

Montroll [14] considered N lattice sites that could be arranged on a line, in a square or in a cube. His interest was to determine an optimal configuration for an exciton created at one of the N sites, by the absorption of a photon, to randomly hop to a reaction center situated at one of the other sites. This is the initial step in the photosynthesis process. Montroll used generating function techniques and he calculated the average number of nearest-neighbor steps, $\langle n \rangle$, before trapping to be

$$\langle n \rangle \approx \begin{cases} N^2/6 & 1d \\ \frac{1}{\pi} N \log N & 2d \\ 1.5164N & 3d. \end{cases} \quad (7)$$

For N roughly less than 10^5 the $2d$ configuration is the most efficient for trapping the exciton. The photosynthetic unit has N less than 1000.

6. Why Lévy [17]?

For the trapping problem, consider a random walker with diffusion constant D and with mean square displacement after n steps of $\langle R^2(n) \rangle \sim Dn$. Now add traps (absorbing sites) periodically with one trap per N lattice sites. In photosynthesis the trapping site was an absorbing site. In a search for food the trap could represent a food source. On average it will take the random walker about N jumps to become trapped and the lifetime distribution function $f(t)$ will be

$$f(t) = \exp(-\text{const } Dt/R^2). \quad (8)$$

But what if the traps are introduced at random positions [15]. On average there will still be one trap per N sites, but the possibility arises that some regions larger than N sites will have no traps and other regions will have more than one trap. In fact, there will be a Poisson distribution of volumes without traps, i.e. the probability of finding such a trapless volume V is given by

$$p(V) = \frac{1}{V_0} \exp(-V/V_0). \quad (9)$$

In general, $V \approx R^d$ in d dimensions. The probability of survival $\varphi(\tau)$ through time t is now given by

$$\varphi(t) \approx \int_0^\infty \exp(-R^d/V_0) \exp(-\text{const } t/R^2) dR \quad (10)$$

and using a steepest descent analysis this integral's long time tail is seen to be [16]

$$\lim_{t \rightarrow \infty} \varphi(t) \approx \exp\left(-t^{\frac{d}{d+2}}\right). \quad (11)$$

If instead of diffusion the random walker does a Lévy walk at a constant velocity, such that

$$\langle R^2(t) \rangle \propto t^2, \quad (12)$$

then equation (10) becomes

$$\varphi(t) \approx \int_0^\infty \exp(-R^d/V_0) \exp(-\text{const } t/R) dR \quad (13)$$

with the asymptotic behavior of

$$\lim_{t \rightarrow \infty} \varphi(t) \approx \exp\left(-t^{\frac{d}{d+1}}\right), \quad (14)$$

a faster relaxation time distribution.

If being trapped is equated to finding a target (food, treasure, etc), then the Lévy walk beats Brownian motion because one escapes a large volume V devoid of traps more readily with a Lévy trajectory than with a Brownian trajectory. This was the point of our 1986 article [17].

7. Fixed time searches

Suppose one alternates two search modes λ and γ . For an animal this might be walking alternating with flying. Or it might be a bird flying over land and then over water. Let us now limit the search time to a duration T split between the λ search followed by the γ search. The question is how to divide the time between the two search modes to optimize the finding of targets. In mode λ , let the probability of finding a target in a time t be $1 - \exp(-\lambda t)$ if the target is present, and let a target be present with probability α . In mode γ let the related

probabilities be $1 - \exp(-\gamma t)$ and β . Assume that time $T - \tau$ is spend in mode λ and time τ in mode γ ; then the expected number $S(T)$ of target encounters is

$$S(\tau; T) = \alpha(1 - \exp(-\lambda(T - \tau))) + \beta(1 - \exp(-\gamma\tau)), \quad (15)$$

where T is fixed. To find the optimal value of τ we set

$$\frac{\partial S(T; \tau)}{\partial \tau} = 0,$$

yielding

$$\tau = \frac{1}{\lambda + \gamma} \ln \left(\frac{\gamma \beta}{\lambda \alpha} \exp(\lambda T) \right). \quad (16)$$

The value of τ becomes $T/2$ when both modes of search are the same. If $\alpha = \beta$ and $\lambda > \gamma$, then success comes sooner in the λ process so more time should be spend in the γ process, i.e. the optimal $\tau > T/2$. If $\lambda = \gamma$, then $\tau > T/2$ if $\beta > \alpha$, i.e. if the γ region is richer with targets more time should be spent there.

Now let there be n such sequences alternating between the λ and the γ search modes in a time T and the formula for S becomes

$$S(\tau; T, n) = n \left[\alpha \left(1 - \exp \left(-\lambda \left(\frac{T - \tau}{n} \right) \right) \right) + \beta \left(1 - \exp \left(-\gamma \frac{\tau}{n} \right) \right) \right]. \quad (17)$$

For the exponential form of the probability it pays to maximize the number n of such sequences. A cost factor must be added for switching modes to determine an optimal switching rate.

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